HORIZONTAL DEHN SURGERY AND GENERICITY IN THE CURVE COMPLEX

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Abstract. We introduce a general notion of "genericity" for countable subsets of a space with Borel measure, and apply it to the set of vertices in the curve complex of a surface Σ, interpreted as subset of the space of projective measured laminations in Σ, equipped with its natural Lebesgue measure.

We prove that, for any 3-manifold M, the set of curves c on a Heegaard surface $\Sigma \subset M$, such that every non-trivial Dehn twist at c yields a Heegaard splitting of high distance, is generic in the set of all essential simple closed curves on Σ.

Our definition of "genericity" is different and more intrinsic than alternative such existing notions, given e.g. via random walks or via limits of quotients of finite sets.

1. Introduction

For any compact connected orientable surface Σ of genus $g \geq 2$ the *curve complex*, denoted by $\mathcal{C}(\Sigma)$, is a locally infinite simplicial complex, where every m-simplex is determined by a collection of $m+1$ isotopy classes of pairwise disjoint simple closed essential curves on Σ. Every handlebody W with boundary identification map $h : \partial W \xrightarrow{\approx} \Sigma$ determines a *disk complex* $\mathcal{D}(W) \subset \mathcal{C}(\Sigma)$, which is defined by the condition that every curve representing a vertex of $\mathcal{D}(W)$ must bound a disk in W.

For any simple closed essential curve $c \subset \Sigma$ we denote by δ_c the Dehn twist at c. One can perturb the identification map $h : \partial W \xrightarrow{\approx} \Sigma$ by composing it with a power δ_c^m, to get a handlebody W_c^m which is a homeomorphic copy of W (via some $g : W_c^m \xrightarrow{\approx} W$) and has $\delta_c^m \circ h \circ g : \partial W_c^m \to \Sigma$ as the induced boundary identification map.

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Any Heegaard splitting $M = V \underset{\partial V \approx \Sigma \approx \partial W}{\cup} W$ of a compact closed orientable 3-manifold M defines two subcomplexes $\mathcal{D}(V), \mathcal{D}(W) \subset \mathcal{C}(\Sigma)$, and the minimal distance (in the simplicial metric of $\mathcal{C}(\Sigma)$) between points in these subcomplexes is called the *distance* of the Heegaard splitting, denoted $d(V, W)$.

Thus any simple closed essential curve $c \subset \Sigma \subset M$ and any integer $m \in \mathbb{Z}$ define a new 3-manifold M_c^m with Heegaard splitting:

$$M_c^m = V \underset{\partial V \approx \Sigma \approx \partial W_c^m}{\cup} W_c^m$$

The curve c is called *n-optimal* if for every non-trivial twist exponent $m \neq 0$ one has:

$$d(V, W_c^m) \geq n$$

The main result of this paper, Theorem 6.8, can be paraphrased as follows:

Theorem 1.1. *Let M be a closed orientable 3-manifold, and let Σ be a Heegaard surface of M. Then almost every essential simple closed curve c on Σ is n-optimal, for any $n \geq 1$.*

Here the terminology "almost every" refers to the Lebesgue measure on $\mathcal{PML}(\Sigma)$, the space of projective measured laminations on Σ. Typically, genericity results come from random walks and almost always involve at the very beginning the choice of extra data, for example a preferred generating system of a group (see e.g. [5], [17]). Generic sets tend to vary if one modifies these extra data.

Alternatively, a "complexity" function with finite preimage subsets is used, and genericity is defined by stating that the quotient of the cardinalities of certain sets of bounded complexity tends to 1 if the complexity bound tends to ∞ (see e.g. [1], [23], [14]). Again, such genericity results depend heavily on the choice of the complexity function at the very beginning.

The concept of *genericity* introduced in this paper is independent of any such additional choices and hence is somewhat preferable. We give a detailed discussion of this concept in Section 4.1 of this paper.

Remark 1.2. The proof of the above theorem is constructive. For example, in the special case of the 3-sphere $M = S^3$ with a standard Heegaard surface Σ of genus g, it exhibits, for any given $n \geq 1$, explicit curves c on Σ with the property that a single Dehn twist at c alters the standard Heegaard splitting of S^3 to a new Heegaard splittings (of the new manifold $S_c^{3,1}$) with distance bigger or equal to n. Notice also that the manifolds $S_c^{3,m}$ can alternatively be obtained by $\frac{1}{m}$-surgery on the

knot $c \subset S^3$, where the surgery coefficients are defined with respect to the meridian on $\partial N(c)$ and the "horizontal" slope $\Sigma \cap \partial N(c)$. Manifolds M_c^m defined as above are said to be obtained from M by *horizontal Dehn surgery on the knot* $c \subset \Sigma$ (see e.g. [22]).

The curve complex associated with a closed surface has become over the past ten years a subject of increasing importance for low dimensional topology. It is known to be a δ-hyperbolic space in the sense of Gromov (see [19]). For a 3-manifold M with Heegaard splitting $M = V \cup_\Sigma W$ the above defined distance $d(V, W)$ (sometimes also referred to as the *Hempel distance* of the splitting) has become an important invariant of the Heegaard splitting. For example, it has been shown that any 3-manifold admits only finitely many Heegaard splittings of distance ≥ 3 (see [26]). Furthermore, it follows from Perelman's proof of the Geometrization Conjecture and from the classification of Heegaard splittings of Seifert fibered spaces (see [22] and [9]) that every 3-manifold M with at least one Heegaard splitting of distance ≥ 3 is hyperbolic.

In the process of proving the above results, we have also derived the following two genericity statements about distance in the curve complex, which may be of interest in their own right. Since they confirm what most experts feel ought to be true, they can alternatively be viewed as confirmation that the definition of "genericity" introduced in this paper is a useful and natural notion. We have:

Corollary 5.3. For any essential simple closed curve c on Σ, the set $\mathcal{C}_n^0(c)$ of all essential simple closed curves k on Σ with distance

$$d(k, c) \geq n$$

is generic in the set $\mathcal{C}^0(\Sigma)$ of all essential simple closed curves on Σ.

Theorem 5.2. For any handlebody H with boundary surface $\partial H = \Sigma$ the set $\mathcal{C}_n^0(H)$ of all essential simple closed curves k on Σ with distance

$$d(k, H) = \min\{d(k, c) \mid c \in \mathcal{D}(H)\} \geq n$$

is generic in the set $\mathcal{C}^0(\Sigma)$.

We would like to point out that an important ingredient in the proofs of the above theorems is Kerckhoff's result that the limit set of the handlebody group has measure zero in the Thurston boundary (see [11]).

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2. Notation and background

In this section we will recall various definitions and background material needed for the following sections. Most of the material of this section has been presented in full detail in [16], and is here only recalled briefly, for the convenience of the reader.

2.1. **The curve complex.**

Given an orientable connected surface Σ of genus $g \geq 2$, the *curve complex*, denoted by $\mathcal{C}(\Sigma)$ is defined as follows:

(1) The set of vertices $\mathcal{C}^0(\Sigma)$ is the set of isotopy classes of simple closed curves on Σ.
(2) An n-simplex in $\mathcal{C}(\Sigma)$ is a collection $\{v_0, ..., v_n\}$ of vertices which can be represented by mutually disjoint curves.

On the 1-skeleton of $\mathcal{C}^1(\Sigma)$ one defines a metric $d_{\mathcal{C}}(\cdot, \cdot)$ by declaring the length of every edge to be 1. For the purpose of this paper it will suffice to consider only $\mathcal{C}^1(\Sigma)$.

2.2. **Train tracks.**

A *train track* τ in Σ is a compact subsurface with boundary, which is equipped with a *singular I-fiberation*: The interior of τ is fibered by open arcs, and the fibration extends to a fiberation of the compact surface τ by properly embedded closed arcs (the *I-fibers*), except for finitely many *singular points* (also called *cusp points*) on $\partial\tau$, where precisely two fibers meet. We call these fibers *singular fibers*. We admit the case that a fiber is *doubly singular*, i.e. both of its endpoints are singular points.

Two singular fibers are *adjacent* if they share a singular point as a common endpoint. A maximal connected union of singular or doubly singular I-fibers is called an *exceptional fiber*. It is either homeomorphic

[1] Y. Minsky also provided considerable help with the figures.

to a closed interval, or to a simple closed curve on Σ. In the latter case it will be called a *cyclic exceptional fiber*. We explicitely admit this second case, although in the classical train track literature this case is sometimes suppressed.

Definition 2.1. A train track $\tau \subset \Sigma$ is called *fat* if all of its exceptional fibers are cyclic. We denote by $\mathcal{E}_\tau$ the collection of simple closed curves on Σ given by the exceptional fibers of τ.

A train track τ in Σ is called *filling*, if all complementary components of τ in Σ are simply connected. The train track τ is called *maximal*, if every complementary component is a *triangle*, i.e. it is simply connected and there are precisely three singular points on its boundary.

An arc, a closed curve or a lamination in Σ *is carried* by a train track $\tau \subset \Sigma$ if it is contained in τ and is throughout transverse to the I-fibers of τ. Two simple arcs carried by τ are *parallel* if they intersect the same I-fibers, and these intersections occur on the two arcs in precisely the same order. An arc, a closed curve or a lamination on Σ which is carried by τ is said to *cover* τ if it meets every I-fiber of τ.

2.3. **Unzipping paths and derived train tracks.**

Given a train track $\tau \subset \Sigma$ which carries a lamination $\mathcal{L}$ we can obtain a new train track, which still carries $\mathcal{L}$, as follows:

The train track τ can be *split* by moving any of the singular points P (now called a *zipper*), which is located on the boundary of a complementary component Δ of τ, into the interior of τ. The zipper P will move along an *unzipping path*, which is embedded in the interior of $\tau \smallsetminus \mathcal{L}$ and is transverse to the I-fibers. Two unzipping paths are not allowed to cross each other. An unzipping path which covers τ is called *complete*.

In case two zippers meet the same connected component of an I-fiber in $\tau \smallsetminus \mathcal{L}$ from different directions, they have to join up, thus changing the topology of the train track and of its complementary components. A situation like this is called a *collision*. In case of a collision the unzipping procedure stops.

Definition 2.2. We say that τ can be derived with respect to $\mathcal{L}$ if we can successively (or simultaneously, it does not make any difference) unzip every zipper along a complete unzipping path, without ever running into a collision. The train track τ' obtained by unzipping along shortest possible complete unzipping paths is said to be *derived from τ with respect to $\mathcal{L}$*, or simply *derived from τ*.

Remark 2.3. If the train track τ' is derived from a maximal train track τ, then every complementary component Δ of τ' is also a triangle, i.e. τ' is also maximal. This follows directly from Definition 2.2, since during the deriving process the unzipping paths never run into collisions.

Lemma 2.4 ([15], Lemma 2.9)**.** *Given a surface Σ and maximal train tracks $\tau, \tau' \subset \Sigma$ so that τ' is derived from τ. Let D be a simple closed curve carried by τ'. Then D covers τ.*

A collection of train tracks $\tau_0 \supset \tau_1 \supset \ldots \supset \tau_n$ will be called an *n-tower of derived train tracks in Σ* if each τ_i is derived from τ_{i-1}, for all $i = 1, \ldots, n$. In this case we say that τ_n has been *n times derived* from τ_0.

2.4. **Complete fat train tracks.**

A curve D is called *tight* with respect to a system of pairwise disjoint essential simple closed curves $\mathcal{E} = \{E_1, \ldots, E_r\}$ in Σ if the number of intersection points with $\mathcal{E}$ can not be strictly decreased by an isotopy of D. The same terminology is used for arcs α which have their endpoints on $\mathcal{E}$, where the endpoints cannot leave $\mathcal{E}$ throughout the isotopy.

Definition 2.5. Let $P \subset \Sigma$ be a pair-of-pants, i.e. a sphere with three open disks removed.

(a) A simple arc in P which has its two endpoints on different components of ∂P will be called a *seam*.
(b) A simple arc in P which has both endpoints on the same component of ∂P, and is not ∂-parallel, will be called a *wave*.
(c) An essential simple closed curve $D \subset \Sigma$ has a wave (or a seam) with respect to a system of curves $\mathcal{E} \subset \Sigma$ if D is tight with respect to $\mathcal{E}$ and if it contains a subarc that is a wave (or a seam) in a complementary component P_i of $\mathcal{E}$ in Σ which is a pair-of-pants.
(d) An essential simple closed curve $D \subset \Sigma$ has a *wave* with respect to a fat train track τ if D has a wave with respect to $\mathcal{E}_\tau$, or if D is isotopic to some $E_k \in \mathcal{E}_\tau$.

A system $\mathcal{E}$ of pairwise disjoint essential simple closed curves on Σ is called a *complete decomposing system* if every complementary component of $\mathcal{E}$ in Σ is a pair-of-pants.

Definition 2.6. A fat train track $\tau \subset \Sigma$ is called *complete* if the following conditions are satisfied:

(1) The collection $\mathcal{E}_\tau$ of exceptional fibers of τ is a complete decomposing system on Σ.
(2) Each pair-of-pants P_i complementary to the system $\mathcal{E}_\tau$ contains two triangles as complementary components of τ in P_i.
(3) The train track τ only carries seams, but no waves, with respect to the complete decomposing system $\mathcal{E}_\tau$.

Notice that every complete fat train track is in particular maximal.

Remark 2.7. Let $\mathcal{E}$ be a complete decomposing system on the surface Σ, and let D be an essential simple closed curve (or a system of such curves) on Σ that is tight with respect to $\mathcal{E}$. We say that D *fills* a pair-of-pants P complementary to $\mathcal{E}$, if $D \cap P$ is the disjoint union of precisely 3 distinct isotopy classes of intersection arcs. Then the following three statements are equivalent:

(1) The curve D fills every pair-of-pants complementary to $\mathcal{E}$, and none of the intersection arcs is a wave.
(2) There exists a unique complete fat train track τ with exceptional fibers $\mathcal{E}_\tau = \mathcal{E}$ that carries D.
(3) There exists some complete fat train track τ with exceptional fibers $\mathcal{E}_\tau = \mathcal{E}$ that is covered by D.

Lemma 2.8 (Lemma 3.5 of [16])**.** *Let $\mathcal{E} \subset \Sigma$ be a complete decomposing system. Any essential simple closed curve $D \subset \Sigma$ which does not have a wave with respect to $\mathcal{E}$, and is not parallel to any $E_k \in \mathcal{E}$, is carried by some complete fat train track τ with exceptional fibers $\mathcal{E}_\tau = \mathcal{E}$.*

The same is true for any system $\mathcal{D}$ of pairwise disjoint essential simple closed curves which satisfy the same conditions as D, or any tight lamination $\mathcal{L}$.

2.5. **Tight train tracks.**

Definition 2.9. A maximal train track τ is called *tight* with respect to some complete decomposing system $\mathcal{E}$ on Σ if the following conditions are satisfied:

(1) For any curve $E_k \in \mathcal{E}$ every connected component of $E_k \cap \tau$ is a disjoint union of (possibly exceptional) I-fibers of τ.
(2) For every connected component Δ_j complementary to τ the intersection segments with any $E_k \in \mathcal{E}$ are arcs with endpoints on two distinct sides of Δ_j.

(3) Each of the three cusps of any complementary component Δ_j is contained in some of the E_k.

The condition (3) of Definition 2.9 is equivalent to stating that every singular I-fiber of τ lies on some of the curves $E_i \in \mathcal{E}$.

The reader may want to note that the above definition is less restrictive than what it appears: A train track which is tight with respect to $\mathcal{E}$ may well carry a wave with respect to $\mathcal{E}$!

Lemma 2.10. *If a maximal train track τ is obtained from deriving finitely many times some fat train track $\widehat{\tau}$ on Σ with $\mathcal{E}_{\widehat{\tau}} = \mathcal{E}$, then τ is tight with respect to $\mathcal{E}$.*

Proof. It follows directly from the definition of a fat train track (see Definition 2.4) that $\widehat{\tau}$ satisfies all three conditions of Definition 2.9. On the other hand, one verifies directly that conditions (1) and (2) are preserved in the unzipping process.

In order to prove condition (3), we assume by induction that it is satisfied by the train track τ' from which τ is derived. By Definition 2.2 for each cusp point Z of a complementary component Δ_j of τ there is a cusp point Z' on the complementary component $\Delta'_j \subset \Delta_j$ of τ' and an unzipping path σ_i that starts at Z' and ends in Z. Recall that by definition of the deriving process no unzipping path can contain a proper initial subpath that meets every I-fiber of τ'.

Now, since σ_i is transverse to the I-fibering of τ', every time that σ_i traverses an I-fiber I_m of τ' contained in some E_k, there is a well defined adjacent (possibly exceptional) I-fiber I_{m+1} contained in some E_l which must be traversed next by σ_i: This fact follows from our inductive hypothesis, which implies (via property (3) of Definition 2.9) that every singular fiber of τ' is contained in some E_h.

As a consequence, if some I-fiber I' of τ' between I_m and I_{m+1} has been traversed by a proper subpath of σ_i, then the same must be true for any other I-fiber I'' between I_m and I_{m+1}. It follows that the endpoint Z of σ_i must be a singular fiber of τ', which has been shown above to lie on some E_k.

□

Proposition 2.11. *Let $\mathcal{E}$ be a complete decomposing system of Σ, and let τ be a maximal train track that is tight with respect to $\mathcal{E}$. Let c be a simple closed curve (or a finite collection of such) on Σ that is tight with respect to $\mathcal{E}$ and contains a subarc β which covers τ. Then c can be carried by τ.*

Proof. We first consider the cyclic sequence of intersection points of c with the singular fibers of τ, and the segments c_i between any two of them, to get a decomposition of c as cyclic concatenation of the c_i. If any of the c_i is contained in τ, we may assume after an inessential isotopy that c_i is carried by τ, while keeping c tight with respect to $\mathcal{E}$. Since τ is tight with respect to $\mathcal{E}$, it follows from conditions (1) and (3) of Definition 2.9 that every singular fiber of τ is a subarc of some $E_i \in \mathcal{E}$. Hence it suffices to show that any of the c_i not contained in τ is contained in a possibly larger segment c_i' of c with endpoints $\partial c_i'$ on $\mathcal{E}$, and that c_i' can be moved into τ by an isotopy relative to its endpoints, while keeping it tight with respect to $\mathcal{E}$. This will be done now by considering one-by-one each of the complementary components Δ of τ and moving c off Δ.

Since β covers τ, it follows from Lemma 2.4 of [16] that for every complementary component Δ of τ, with sides ,say, $\delta_1, \delta_2, \delta_3$, there is for any $i = 1, 2, 3$ a subpath δ_i' of β that runs parallel to all of δ_i. Let I_1, I_2 and I_3 be the singular I-fibers which contain the cusp points of Δ. Let $\widehat{\Delta}$ be the "hexagon" with sides alternatingly situated on one of the δ_i' or on one of the I_k, and which contains Δ.

Since c is simple, any connected component c' of $c \cap \widehat{\Delta}$ must have endpoints on some I_j and I_k. From the tightness of c it follows $I_j \neq I_k$, since both of them are contained in curves from $\mathcal{E}$, by conditions (1) and (3) of Definition 2.9.

Thus, as $\widehat{\Delta}$ is simply connected, c' can be isotoped to an arc c'' that is contained in one of three connected components Δ^i of $\widehat{\Delta} \smallsetminus \Delta$, and this can be done simultaneously with all such arcs c' while keeping the curve c simple, and also keeping it tight with respect to $\mathcal{E}$, by condition (2) of Definition 2.9. But each Δ^i belongs to τ, and we can assume that c'' is transverse to the I-fibers.

It follows, after performing all of these isotopies for any complementary component Δ of τ, that c is carried by τ.

$\square$

Remark 2.12. The statement of Proposition 2.11 includes that of Lemma 3.9 of [16]. Unfortunately it seems that the proof given there is not quite correct; hence the above proof serves also as correction of the latter.

2.6. **A distance criterion in the curve complex.**

Sequences of nested train tracks, as given in the previous definition, occur already in [19], Section 3.1, where they are used to derive lower

bounds for the distance in the curve complex. Indeed, the following statement is a variant of their "Basic observation". A detailed proof is given in [16], stated there as Proposition 2.12 and Remark 2.13.

Corollary 2.13. *For $n \geq 1$ let $\tau_0 \supset \tau_1 \supset \ldots \supset \tau_n$ be an n-tower of derived train tracks in Σ. Assume that τ_0 is a complete fat train track. Let D be an essential simple closed curve carried by τ_n, and let E be an essential simple closed curve which has a wave with respect to τ_0. Then one has:*

$$d_{\mathcal{C}}(D, E) \geq n + 1$$

2.7. **Heegaard splittings.**

Let H be a handlebody of genus $g \geq 2$, and let $\Sigma = \partial H$ denote its boundary surface. The set $\mathcal{D}(H)$ of isotopy classes of essential simple closed curves on Σ that bound a disk in H is a subset of $\mathcal{C}^0(\Sigma)$. It is the vertex set of what is called the *disk complex* of the handlebody H, contained as a subcomplex in $\mathcal{C}(\Sigma)$.

Similarly, we consider complete decomposing systems, up to isotopy in Σ, which bound disk systems in H, and denote the set of such isotopy classes by $\mathcal{CDS}(H)$.

Any closed orientable 3-manifold M has a *Heegaard splitting*, which is a decomposition of M along a surface Σ into two genus g handlebodies V and W, so that $M = V \cup_\Sigma W$. The genus of the *Heegaard surface* Σ is called the *genus* of the Heegaard splitting.

The *distance* of a Heegaard splitting $M = V \cup_\Sigma W$ is defined by

$$d(V, W) = \min\{d_{\mathcal{C}}(D, E) \mid D \in \mathcal{D}(V), E \in \mathcal{D}(W)\},$$

where $d_{\mathcal{C}}$ denotes, as before, the distance in the curve complex $\mathcal{C}(\Sigma)$ (see [9]).

Remark 2.14. Given a complete decomposing system

$$\mathcal{D} = \{D_1, ..., D_{3g-3}\} \in \mathcal{CDS}(V)$$

for a handlebody V of genus $g \geq 2$, then any other essential disk-bounding curve $D \in \mathcal{D}(V)$ is either parallel to one of D_i, or D has a wave with respect to $\mathcal{D}$.

A complete decomposing system $\mathcal{D} = \{D_1, ..., D_{3g-3}\} \subset \Sigma$ is said to *have a wave with respect to a second complete decomposing system* $\mathcal{E} \subset \Sigma$ if some of the D_i has a wave with respect to $\mathcal{E}$.

Lemma 2.15 ([9], Lemma 1.3)**.** *For every Heegaard splitting of a 3-manifold $M = V \cup_\Sigma W$ there always exists a pair of complete decomposing systems $\mathcal{D} \in \mathcal{CDS}(V)$ and $\mathcal{E} \in \mathcal{CDS}(W)$ which have no waves with respect to each other.*

The following is the main result of [16]:

Theorem 2.16 (Theorem 4.7 of [16])**.** *Let M be an oriented 3-manifold with a Heegaard splitting $M = V \cup_\Sigma W$. Consider complete decomposing systems $\mathcal{D} \in \mathcal{CDS}(V)$ and $\mathcal{E} \in \mathcal{CDS}(W)$ which do not have waves with respect to each other. Let $\tau \subset \Sigma$ be a complete fat train track with exceptional fibers $\mathcal{E}_\tau = \mathcal{E}$, and assume that $\mathcal{D}$ is carried by τ_n, for some n-tower of derived train tracks $\tau = \tau_0 \supset \tau_1 \supset \ldots \supset \tau_n$ with $n \geq 2$. Then the distance of the given Heegaard splitting satisfies:*

$$d(V, W) \geq n$$

3. Gregarious laminations

A lamination $\mathcal{L} \subset \Sigma$ is called *minimal* if every leaf is dense in $\mathcal{L}$. A lamination $\mathcal{L} \subset \Sigma$ is called *filling* if the components of $\Sigma \smallsetminus \mathcal{L}$ are simply connected. A *measured geodesic lamination* $(\mathcal{L}, \mu)$ on Σ is a lamination $\mathcal{L}$ together with a transverse measure μ supported on $\mathcal{L}$ (see [24]). Such a measured lamination $(\mathcal{L}, \mu)$ is called *uniquely ergodic* if any transverse measure supported on $\mathcal{L}$ is a multiple of μ. As is common use, we denote the space of projective measured laminations $[\mathcal{L}, \mu]$ on Σ by $\mathcal{PML}(\Sigma)$ (see [7] and [11]). It comes with a natural measure class given by Thurston's p.l.-structure of the $(6g-7)$-dimensional sphere $\mathcal{PML}(\Sigma)$.

Recall that a train track τ is called maximal if every connected component of $\Sigma \smallsetminus \tau$ is a triangle.

Lemma 3.1. *The subset of $\mathcal{PML}(\Sigma)$ given by all minimal laminations has full measure in $\mathcal{PML}(\Sigma)$.*

Proof. It is well known (see [18], [27]) that the set of uniquely ergodic laminations has full measure in $\mathcal{PML}(\Sigma)$. We only need to consider geodesic laminations $\mathcal{L}$ that are given as the support of some transverse measure μ carried by $\mathcal{L}$. Since every such lamination which is not minimal is also non-uniquely ergodic, the set of minimal filling laminations contains the uniquely ergodic ones, which shows the desired conclusion. □

Any uniquely ergodic measured lamination $(\mathcal{L}, \mu)$ has the property that the geodesic lamination $\mathcal{L}$ determines the corresponding projective class $[\mathcal{L}, \mu] \in \mathcal{PML}(\Sigma)$. This justifies a certain amount of sloppyness in suppressing the difference between laminations and projective classes of measured laminations.

Definition 3.2. Let $\tau \subset \Sigma$ be a train track. We will use the following notation:

(1) $\mathcal{PML}(\tau)$ is the set of projective measured laminations carried by τ.

(2) $\mathbf{P}(\tau) \subset \mathcal{PML}(\tau)$ is the subset of all projective measured laminations which are carried by τ and have positive weights on every edge of τ. Such a lamination will be called τ-positive.

(3) $\mathbf{M}(\tau) \subset \mathcal{PML}(\tau)$ denotes the subset given by all minimal laminations which are carried by τ. Set $\mathbf{MP}(\tau) = \mathbf{M}(\tau) \cap \mathbf{P}(\tau)$.

(4) $\mathbf{A}(\tau) \subset \mathcal{PML}(\tau)$ denotes the subset given by all *arational* laminations (i.e. not given by a collection of simple closed curves) which are carried by τ. Set $\mathbf{AMP}(\tau) = \mathbf{A}(\tau) \cap \mathbf{M}(\tau) \cap \mathbf{P}(\tau)$.

To be specific, let us observe here that an element $[\mathcal{L}, \mu]$ of $\mathbf{M}(\tau)$ does not belong to $\mathbf{A}(\tau)$ if and only if the support of μ consists of a single closed curve. If $[\mathcal{L}, \mu] \in \mathbf{M}(\tau) \cap \mathbf{A}(\tau)$, then the support of μ is *totally arational*: it does not contain any closed leaf.

Lemma 3.3. *Let $\tau \subset \Sigma$ be a maximal train track. Then one has:*

(1) *The set $\mathbf{P}(\tau)$ is open in $\mathcal{PML}(\Sigma)$.*

(2) *The set $\mathcal{PML}(\tau)$ has positive measure in $\mathcal{PML}(\Sigma)$.*

(3) *The set $\mathbf{AMP}(\tau)$ has full measure in $\mathcal{PML}(\tau)$.*

Proof. Since τ is maximal, it has only triangles as complementary components. Hence the set $\mathcal{PML}(\tau)$ is a top dimensional cell in $\mathcal{PML}(\Sigma)$ and hence is has positive measure. Furthermore, Lemma 3.1 implies that $\mathbf{M}(\tau)$ has full measure in $\mathcal{PML}(\tau)$. On the other hand, $\mathbf{P}(\tau)$ is precisely the interior of this top dimensional cell, and hence it is open and has full measure in $\mathcal{PML}(\tau)$. Thus $\mathbf{MP}(\tau) = \mathbf{M}(\tau) \cap \mathbf{P}(\tau)$ has full measure in $\mathcal{PML}(\tau)$. But the set of rational laminations is countable and hence of measure 0. Thus $\mathbf{AMP}(\tau)$ has also full measure in $\mathcal{PML}(\tau)$.

$\square$

Definition 3.4. Let τ be a train track on Σ.

(a) A lamination $\mathcal{L}$ carried by τ is called *gregarious*[2] with respect to τ if the train track τ can be derived with respect to $\mathcal{L}$ (compare Definition 2.2).

(b) The subset of $\mathcal{PML}(\tau)$ defined by all gregarious laminations with respect to τ will be denoted by $\mathbf{G}(\tau)$. Set $\mathbf{GP}(\tau) = \mathbf{G}(\tau) \cap \mathbf{P}(\tau)$.

Lemma 3.5. *Let τ be a maximal train track on Σ, and let $\mathcal{L}$ be an arational minimal τ-positive lamination. Then $\mathcal{L}$ is gregarious with respect to τ:*

$$\mathbf{AMP}(\tau) \subset \mathbf{GP}(\tau)$$

Proof. This is a direct consequence of Definition 3.4: Since $\mathcal{L}$ is τ-positive it covers τ, and since it is arational, no two zippers can ever meet, for arbitrary long unzipping paths. Since $\mathcal{L}$ is minimal, every unzipping path will eventually intersect every transverse I-fiber which is met by $\mathcal{L}$. But $\mathcal{L}$ is minimal and covers τ, so that the unzipping paths will eventually become complete.

□

The next lemma states that the inclusion from the previous lemma, though an equality "in measure" (by Lemma 3.3 (3)), is proper, since $\mathbf{AMP}(\tau)$ is disjoint from the dense set of laminations supported by a single closed curve.

Lemma 3.6. *Let $\tau \subset \Sigma$ be a maximal train track. Then the set $\mathbf{GP}(\tau)$ is open in $\mathcal{PML}(\tau)$.*

Proof. Let $\mathcal{L}$ be a τ-positive lamination which is gregarious with respect to τ and let $\tau' = \tau(\mathcal{L})$ be the derived train track. Since τ is maximal, it follows that τ' also has only triangles as complementary components. An elementary Euler characteristic count shows that $\mathcal{PML}(\tau')$ is a subcell of maximal dimension in $\mathcal{PML}(\tau)$, and thus any open subset of $\mathcal{PML}(\tau')$ is open in $\mathcal{PML}(\tau)$.

Every element $[\mathcal{L}, \mu]$ of $\mathbf{GP}(\tau) \subset \mathcal{PML}(\tau') \subset \mathcal{PML}(\tau) \subset \mathcal{PML}(\mathbf{\Sigma})$ defines a set of weights on the edges of τ', where we impose the additional condition that the sum of these weights is one: Otherwise the weights would only be determined up to a scalar factor.

Perturbing the weights on the edges of τ' slightly determines an open neighborhood of $[\mathcal{L}, \mu] \in \mathbf{GP}(\tau)$ in $\mathcal{PML}(\tau')$. This neighborhood is

[2] The authors would like to thank Wendy Sandler for suggesting this inspiring terminology.

also open in $\mathcal{PML}(\tau)$, and (by Lemma 3.3 (1)) even in $\mathcal{PML}(\Sigma)$. Since any lamination in this neighborhood is carried by τ', the neighborhood consists entirely of laminations in $\mathbf{GP}(\tau)$. Thus $\mathbf{GP}(\tau)$ is open in $\mathcal{PML}(\tau)$.

□

Lemma 3.7. *Let $\tau \subset \Sigma$ be a maximal train track. Then the set $\mathbf{GP}(\tau)$ has full measure in $\mathcal{PML}(\tau)$.*

Proof. This is a direct consequence of Lemma 3.5 and Lemma 3.3 (3). □

We will now consider the the subset $\mathcal{PML}_0(\Sigma)$ of $\mathcal{PML}(\Sigma)$ which consists of projective measured laminations $[\mathcal{L}, \mu]$ such that $\mathcal{L}$ is a single closed curve. Since a simple closed curve carries up to scalar multiples only one transverse measure, there is a canonical identification $\mathcal{PML}_0(\Sigma) = \mathcal{C}^0(\Sigma)$.

In analogy to $\mathcal{PML}_0(\Sigma)$, we introduce $\mathcal{PML}_0(\tau) = \mathcal{PML}(\tau) \cap \mathcal{PML}_0(\Sigma)$, $\mathbf{GP_0}(\tau) = \mathbf{GP}(\tau) \cap \mathcal{PML}_\mathbf{0}(\mathbf{\Sigma})$, etc. Of course, all these newly introduced sets are countable, and they are dense in their "parent" set, if the latter is open in $\mathcal{PML}(\Sigma)$.

Lemma 3.8. *Let $\tau \subset \Sigma$ be a maximal train track. Then there is a countable family of maximal train tracks $\tau_1, \tau_2, \ldots$, each of which is derived from τ, such that $\mathbf{GP}(\tau)$ is equal to the union of all $\mathbf{P}(\tau_i)$.*

Proof. Since $\mathcal{PML}_0(\Sigma)$ is dense in $\mathcal{PML}(\Sigma)$ and $\mathbf{GP}(\tau) \subset \mathbf{P}(\tau)$ is open in $\mathcal{PML}(\Sigma)$ (by Lemma 3.6 and Lemma 3.3 (1)), it follows that the countable set $\mathbf{GP_0}(\tau)$ is dense in $\mathbf{GP}(\tau)$. Notice that for each element D_i of $\mathbf{GP_0}(\tau)$ there is a maximal train track τ_i derived from τ which carries D_i, and that $\mathbf{P}(\tau_i)$ is an open neighborhood of D_i in $\mathcal{PML}(\tau)$, again by Lemma 3.6. Hence the union of all $\mathbf{P}(\tau_i)$ contains all of $\mathbf{GP}(\tau)$, and hence is equal to the latter.

□

The notion of gregariousness can be strengthened further:

Definition 3.9. Let $\tau \subset \Sigma$ be a train track, and let $\mathcal{L}$ be a lamination carried by τ. We say that $\mathcal{L}$ is *n-gregarious* with respect to τ if τ can be derived n times with respect to $\mathcal{L}$, i.e. there exists a tower

$$\tau = \tau_0 \supset \tau_1 \supset \ldots \supset \tau_n$$

of derived train tracks with respect to $\mathcal{L}$. In particular $\mathcal{L}$ is carried by τ_n. We denote the subset of $\mathcal{PML}(\tau)$ given by all n-gregarious laminations by $\mathbf{G}^n(\tau)$, and define $\mathbf{G^nP}(\tau) = \mathbf{G^n}(\tau) \cap \mathbf{P}(\tau)$.

Proposition 3.10. *Let $\tau \subset \Sigma$ be a maximal train track. Then for all $n \geq 1$ the set $\mathbf{G^nP}(\tau)$ of n-gregarious laminations which are τ-positive is open in $\mathcal{PML}(\tau)$ and in $\mathcal{PML}(\Sigma)$.*

Proof. By Lemma 3.3 (1) it suffices to prove openess in $\mathcal{PML}(\tau)$. Assume by induction that $\mathbf{G^{n-1}P}(\tau)$ is open in $\mathcal{PML}(\tau)$, and that there is a countable family of maximal train tracks τ_i that are obtained from τ by deriving $n-1$ times, such that $\mathbf{G^{n-1}P}(\tau)$ is equal to the union of all $\mathbf{P}(\tau_i)$. Thus $\mathbf{G^nP}(\tau)$ is equal to the union of all $\mathbf{GP}(\tau_i)$.

Now apply Lemma 3.8 to each of the τ_i to get a countable family of maximal train tracks $\tau_{i,j}$ derived from τ_i, such that $\mathbf{GP}(\tau_i)$ is equal to the union of all $\mathbf{P}(\tau_{i,j})$, for fixed i. It follows that $\mathbf{G^nP}(\tau)$ is equal to the union of the $\mathbf{P}(\tau_{i,j})$, for all i and j.

From Lemma 3.6 we obtain that every $\mathbf{P}(\tau_{i,j})$ is open in $\mathcal{PML}(\tau_{i,j})$. Since all $\tau_{i,j}$ are maximal, the set $\mathcal{PML}(\tau_{i,j})$ is a cell of maximal dimension in $\mathcal{PML}(\tau)$. It follows directly that every $\mathbf{P}(\tau_{i,j})$ is open in $\mathcal{PML}(\tau)$. Hence their union $\mathbf{G^nP}(\tau)$ is also open in $\mathcal{PML}(\tau)$. This completes the induction and hence the proof.

□

Proposition 3.11. *Let $\tau \subset \Sigma$ be a maximal train track. Then for all $n \geq 1$ the set $\mathbf{G^nP}(\tau)$ of n-gregarious laminations which are τ-positive is of full measure in $\mathcal{PML}(\tau)$.*

Proof. By Lemma 3.3 (3) we know that $\mathbf{G^0P}(\tau) = \mathbf{P}(\tau)$ has full measure in $\mathcal{PML}(\tau)$. Hence the claim follows by induction if one proves that $\mathbf{G^{k+1}P}(\tau)$ has full measure in $\mathbf{G^kP}(\tau)$.

Let us first recall that $\mathbf{G^kP}(\tau)$ is the countable union of sets $\mathbf{P}(\tau_i)$, where each τ_i is a maximal train track obtained from τ by deriving k times. This has been shown by induction in the proof of Proposition 3.10.

For any τ_i the set $\mathbf{GP}(\tau_i)$ has full measure in $\mathbf{P}(\tau_i) \subset \mathcal{PML}(\tau_i)$ (by Lemma 3.7). Hence the union of all $\mathbf{GP}(\tau_i)$ has full measure in the union of all $\mathbf{P}(\tau_i)$. But the union of all $\mathbf{GP}(\tau_i)$ is (by definition of the τ_i) equal to $\mathbf{G^{k+1}P}(\tau)$, while the union of all $\mathbf{P}(\tau_i)$ is precisely $\mathbf{G^kP}(\tau)$. This proves the inductive step. □

Proposition 3.12. *For any complete fat train track τ in Σ the set $\mathbf{G_0^n}(\tau)$ has closure $\overline{\mathbf{G_0^n}(\tau)}$ in $\mathcal{PML}(\tau)$ which is of full measure. The complement of $\mathbf{G_0^n}(\tau)$ in $\mathcal{PML}_0(\tau)$ has closure $\overline{\mathcal{PML}_0(\tau) \smallsetminus \mathbf{G_0^n}(\tau)}$ in $\mathcal{PML}(\tau)$ which is of measure 0.*

Proof. The set $\mathcal{PML}_0(\Sigma)$ is dense in $\mathcal{PML}(\Sigma)$, and thus, since $\mathbf{G^nP}(\tau)$ is open in $\mathcal{PML}(\Sigma)$ (by Proposition 3.10), it follows that $\mathbf{G^nP_0}(\tau) = \mathcal{PML}_\mathbf{0}(\mathbf{\Sigma}) \cap \mathbf{G^nP}(\tau)$ is dense in $\mathbf{G^nP}(\tau)$. But by Proposition 3.11 the set $\mathbf{G^nP}(\tau)$ is of full measure in $\mathcal{PML}(\tau)$. Since $\mathbf{G^nP_0}(\tau)$ is a subset of $\mathbf{G^n_0}(\tau)$, it follows that $\overline{\mathbf{G^n_0}(\tau)}$ has full measure in $\mathcal{PML}(\tau)$.

The complementary set $\mathcal{PML}_0(\tau) \smallsetminus \mathbf{G^n_0}(\tau)$ is contained in $\mathcal{PML}_0(\tau) \smallsetminus \mathbf{G^nP_0}(\tau)$, and hence in $\mathcal{PML}(\tau) \smallsetminus \mathbf{G^nP}(\tau)$, since $\mathbf{G^nP}(\tau) \cap \mathcal{PML}_0(\tau) = \mathbf{G^nP_0}(\tau)$. By Propositions 3.10 and 3.11, the set $\mathcal{PML}(\tau) \smallsetminus \mathbf{G^nP}(\tau)$ is a closed set of measure 0 in $\mathcal{PML}(\tau)$.

□

4. Genericity

Classically, a subset of a countable set is called "generic" if its complement is finite. This notion, however, often doesn't capture the geometry of the given set-up.

For example, consider the countable set S of points in the unit square I^2 which have rational coordinates. The subset of S which lies in the interior of I^2 has infinite complement, but everyone will agree that a "generic" point of S will lie in the interior of I^2 and not on its boundary.

In order to address the above problem we propose the following more subtle definition for genericity:

Definition 4.1. Let X be a topological space, provided with a Borel measure μ. Let $Y \subset X$ be a (possibly countable) subset, which is a disjoint union $Y = A \dot\cup B$. The set A is called *generic in* Y (or simply *generic*, if $Y = X$) if the closure $\overline{A}$ of A has measure $\mu(\overline{A}) > 0$, and the closure $\overline{B}$ of B has measure $\mu(\overline{B}) = 0$. To be specific, both closures $\overline{A}$ and $\overline{B}$ are taken in X.

Remark 4.2. Notice that it follows from Definition 4.1 that in the above setting, whether or not a set A is generic, depends only on the measure class of μ, and not on the measure μ itself.

Notice that in this definition the sets $\overline{A}$ and $\overline{B}$ may well not be disjoint, although A and B are assumed to be disjoint. Note also, that this definition of genericity extends to sets Y that are not embedded but are just mapped to X, by a properly chosen "natural" map. It is important to remember that every statement about genericity always depends on a previous choice of a measure. This choice is, formally speaking, arbitrary, and thus can at best be natural.

The following is an immediate consequence of the above definition:

Lemma 4.3. *Given sets X, Y, A and $\overline{A}$ as in Definition 4.1, and let $\overline{Y}$ be the closure of Y in X. Then A is generic in Y if and only if one of the following two equivalent conditions is satisfied:*

(1) *The set $\overline{A}$ contains a set Z which is open in $\overline{Y}$ and of full measure $\mu(Z) = \mu(\overline{Y}) > 0$, and which is disjoint from $Y \smallsetminus A$.*
(2) *The set $\overline{Y}$ has measure $\mu(\overline{Y}) > 0$, and $Y \smallsetminus A$ is contained in a closed set of measure 0.*

□

Remark 4.4. As a direct consequence of Definition 4.1 and its reformulations in Lemma 4.3 we obtain the following:

(a) Arbitrary unions and finite intersections of sets A_i that are generic in a common set $Y \subset X$ are again generic in Y.
(b) For any sets $A \subset A' \subset Y \subset X$, if A is generic in Y, then so is A'.

The situation becomes more complicated if one also varies the set Y.

Proposition 4.5. *Let X, Y and A be as in Definition 4.1. Assume that Y contains a countable union of subsets Y_i, and define $A_i = Y_i \cap A$. Denote, as before, by $\overline{Y}$ and $\overline{Y}_i$ the closures (in X) of Y and Y_i respectively, and assume furthermore that:*

(a) $Y_i = \overline{Y}_i \cap Y$,
(b) $\mu(\overline{Y} \smallsetminus \cup \overline{Y}_i) = 0$, *and*
(c) *every $\overline{Y}_i$ contains some set $\mathring{Y}_i$ that is open in $\overline{Y}$ and has full measure in $\overline{Y}_i$.*

If every A_i is generic in Y_i, then A is generic in Y.

Proof. By assumption every A_i is generic in Y_i. Hence Lemma 4.3 (1) gives sets $Z'_i \subset \overline{A}_i$ that are open in $\overline{Y}_i$, are of full positive measure in $\overline{Y}_i$, and satisfy $Z'_i \cap Y_i \subset A_i$. Define $Z_i = Z'_i \cap \mathring{Y}_i$, and observe that the Z_i are still of full positive measure in $\overline{Y}_i$, and in addition they are open in $\overline{Y}$. Their union $Z = \cup Z_i$ is an open set (in $\overline{Y}$) of positive measure contained in $\overline{A}$, so that $\mu(\overline{A}) > 0$.

The complementary set $\overline{Y} \smallsetminus Z$ is closed in $\overline{Y}$ and hence in X, and it contains $Y \smallsetminus A$, since for any index i one has $Z_i \subset \overline{Y}_i$, and $Z_i \cap Y \subset Z'_i \cap Y = Z'_i \cap \overline{Y}_i \cap Y = Z'_i \cap Y_i \subset A_i$. But $\overline{Y} \smallsetminus Z$ is contained in

$$(\overline{Y} \smallsetminus \cup \overline{Y}_i) \ \cup \ (\cup (\overline{Y}_i \smallsetminus Z_i)) \,,$$

which is a countable union of measure 0 sets. Thus $\overline{Y} \smallsetminus Z$ is of measure 0, which implies that $\overline{Y \smallsetminus A}$ is of measure 0. This proves that A is generic in Y.

$\square$

The following proposition will not be used below, but we believe it can be a useful tool in other contexts.

Proposition 4.6. *Let X and Y be as in Definition 4.1. Assume that the set $A \subset Y$ is given as (a not necessarily disjoint) union of countably many subsets A_i, and for each A_i there are sets X_i and Z_i such that for any index i the following holds:*

1. $Z_i \subset \overline{A}_i \subset X_i \subset X$
2. *X_i is closed, and Z_i is open in X.*
3. *The union of all X_i contains Y.*
4. $Z_i \cap Y \subset A_i$.
5. *Z_i has full measure in X_i.*
6. *Some X_j has positive measure in X.*
7. *$\overline{\cup X_i} \smallsetminus \cup X_i$ has measure 0.*

Then A is generic in Y.

Proof. By (1) Z_i is contained in $\overline{A}$, and by (5) and (6) some Z_i has positive measure. Thus $\overline{A}$ has positive measure. On the other hand, one has:

$$\overline{\bigcup_i X_i} \smallsetminus \bigcup_i Z_i \ \subset \ \overline{\bigcup_i X_i} \smallsetminus \bigcup_i X_i \ \cup \ \bigcup_i X_i \smallsetminus \bigcup_i Z_i$$

and

$$\bigcup_i X_i \smallsetminus \bigcup_i Z_i \ \subset \ \bigcup_i (X_i \smallsetminus Z_i)$$

The set $\bigcup_i (X_i \smallsetminus Z_i)$ has measure 0, by (5) and by the countability of the index set. Similarly, the set $\overline{\bigcup_i X_i} \smallsetminus \bigcup_i X_i$ has measure 0, by assumtion (7). The set $\overline{\bigcup_i X_i} \smallsetminus \bigcup_i Z_i$ is closed, by (2). But by (3) and (4) the set $Y \smallsetminus A$ is contained in $\overline{\bigcup_i X_i} \smallsetminus \bigcup_i Z_i$, so that the closure $\overline{Y \smallsetminus A}$ must have measure 0. Thus A is generic in Y.

$\square$

5. Genericity of large distance in the curve complex

For any handlebody H with boundary $\partial H = \Sigma$ and any integer $n \in \mathbb{N}$ we say that a curve $c \in \mathcal{C}^0(\Sigma) = \mathcal{PML}(\Sigma)$ is *n-gregarious with respect to H*, if c is n-gregarious with respect to some complete fat train track τ with exceptional fibers $\mathcal{E}_\tau$ in $\mathcal{CDS}(H)$ (compare with Definition 3.9). In the terminology of Definition 6.6, this is equivalent to stating that c is n-gregarious with respect to any complete decomposing system $\mathcal{E} \in \mathcal{CDS}(H)$.

Proposition 5.1. *For any integer $n \geq 1$ and any handlebody H with boundary surface $\partial H = \Sigma$ the set $\mathbf{G_0^n}(H)$ of n-gregarious curves c with respect to H is generic in $\mathcal{C}^0(\Sigma)$.*

Proof. We will use Proposition 4.5, with $X = \mathcal{PML}(\Sigma), Y = \mathcal{C}^0(\Sigma) = \mathcal{PML}_0(\Sigma), A = \mathbf{G_0^n}(H)$, and $Y_i = \mathcal{PML}_0(\tau_i)$, where τ_i is any complete fat train track with $\mathcal{E}_{\tau_i} \in \mathcal{CDS}(H)$. Note that the set $\mathcal{CDS}(H)$ of isotopy classes of complete decomposing systems in H is countable, and that for each complete decomposing system $\mathcal{E}$ there are only countably many fat train tracks τ with $\mathcal{E}_\tau = \mathcal{E}$ (up to isotopy of the pair $(\Sigma, \mathcal{E})$). Note also that $\mathcal{PML}_0(\tau_i)$ is dense in $\mathcal{PML}(\tau_i)$, which is closed in $\mathcal{PML}(\Sigma)$, so that $\overline{Y}_i = \mathcal{PML}(\tau_i)$ and $Y_i = \overline{Y}_i \cap Y$ holds.

Furthermore, we know from Lemma 3.3 that the set $\mathbf{P}(\tau_i) \subset \mathcal{PML}(\tau_i)$ is open and full measure in $\mathcal{PML}(\tau_i)$. Since τ_i is maximal, the set $\mathcal{PML}(\tau_i)$ is a cell of maximal dimension in $\mathcal{PML}(\Sigma)$, so that $\mathbf{P}(\tau_i)$ is open in $\overline{Y} = \mathcal{PML}(\Sigma)$. Thus we can define $\overset{\circ}{Y}_i = \mathbf{P}(\tau_i)$.

We now consider the set $\overline{Y} \smallsetminus \cup \overline{Y}_i$: It consists of all laminations $\mathcal{L}$ which are not carried by any complete fat train track with exceptional fibers in $\mathcal{CDS}(H)$. Thus, by Lemma 2.8, $\mathcal{L}$ must have a wave with respect to any complete decomposing system that bounds disks in H. But the set of such laminations $\mathcal{L}$ is precisely the set $\mathcal{R} \subset \mathcal{PML}(\Sigma)$ for which Kerckhoff shows $\mu(\mathcal{R}) = 0$, in his proof that the limit set of the handlebody group has measure 0 (see [11]).

We can now apply Proposition 3.12 to each of the τ_i: It states precisely that $A_i = \mathbf{G_0^n}(\tau_i) = \mathcal{PML}_0(\Sigma) \cap \mathbf{G}^n(\tau_i) = \mathcal{PML}_0(\tau_i) \cap \mathbf{G}^n(H)$ is generic in $Y_i = \mathcal{PML}_0(\tau_i)$. Thus Proposition 4.5 gives the desired conclusion.

$\square$

Denote by $\mathcal{C}^0_n(H) \subset \mathcal{PML}_0(\Sigma)$ the set of essential simple closed curves $D \in \mathcal{PML}_0(\Sigma)$ which satisfy $d_{\mathcal{C}}(D, E) \geq n$ for any $E \in \mathcal{D}(H)$:

$$\mathcal{C}^0_n(H) = \{D \in \mathcal{PML}_0(\Sigma) \mid d_{\mathcal{C}}(D, \mathcal{D}(H)) \geq n\}$$

Recall that any complete fat train track τ on the surface Σ defines a handlebody $H = H(\tau)$ with boundary $\partial H = \Sigma$ by the condition $\mathcal{E}_\tau \in \mathcal{CDS}(H)$, i.e. all $E_i \in \mathcal{E}_\tau$ bound disks in H.

Theorem 5.2. *For every handlebody H with $\partial H = \Sigma$ the set $\mathcal{C}^0_n(H)$ is generic in the set $\mathcal{C}^0(\Sigma)$.*

Proof. For every complete fat train track τ on Σ it follows from Remark 2.14 that every disk in $\mathcal{D}(H(\tau))$ has a wave with respect to τ. Thus it follows from Corollary 2.13 that $\mathcal{C}^0_n(H)$ contains $\mathbf{G}^{\mathbf{n}}_{\mathbf{0}}(H)$. The latter is generic in $\mathcal{C}^0(\Sigma)$, by Proposition 5.1. Thus an application of statement (b) of Remark 4.4 finishes the proof.

□

Corollary 5.3. *For any essential simple closed curve c on Σ, the set $\mathcal{C}^0_n(c)$ of all essential simple closed curves k on Σ with distance*

$$d_{\mathcal{C}}(k, c) \geq n$$

is generic in the set $\mathcal{C}^0(\Sigma)$ of all essential simple closed curves on Σ.

Proof. Consider any handlebody H which contains a disk with boundary curve c, and observe that $c \in \mathcal{D}(H)$ implies

$$\mathcal{C}^0_n(H) \subset \mathcal{C}^0_n(c)\,.$$

Thus statement (b) of Remark 4.4 gives directly the stated claim.

□

6. Intersection and Dehn twists

Let Σ be an orientable surface of genus $g \geq 2$, and let $\mathcal{E}$ be a complete decomposing system for H. Let k be an essential simple closed curve which is tight (see subsection 2.4) with respect to the complete decomposing system $\mathcal{E}$ on Σ. The number of intersection points of k with $\mathcal{E}$ is called the *$\mathcal{E}$-length* of k and is denoted by $\mid k \mid_{\mathcal{E}}$. The same definition and notation will be used for a simple arc α instead of k. However, in this case we always require that $\partial\alpha$ is contained in $\mathcal{E}$, and we count the two points of $\partial\alpha$ as intersection points when we determine the $\mathcal{E}$-length of α.

Two tight simple arcs on Σ are called *parallel* (with respect to $\mathcal{E}$) if they are isotopic to each other via an isotopy of the pair $(\Sigma, \mathcal{E})$. In this case it follows (but this is not equivalent !) that the arcs can be oriented so that their intersections with $\mathcal{E}$ occur at precisely the same sequence of curves $E_j \in \mathcal{E}$, and from the same direction.

We also need to specify what we mean below by a *arc c' on a closed curve c*: Such an arc c' is not necessarily a subarc of c, it can also be the image of a path which is immersed in c but not embedded in c. In particular, c' can wind around c several times.

Let c and k be distinct essential simple closed curves on Σ that are tight with respect to $\mathcal{E}$, and let $P \in c \cap k$ be some intersection point. We now consider maximal parallel arcs α on k and α' on c such that P is contained in $\widehat{\alpha}$ and in $\widehat{\alpha}'$, where $\widehat{\alpha}$ and $\widehat{\alpha}'$ denote the extensions of α on k and α' on c, across the pair-of-pants adjacent to the curves of $\mathcal{E}$ that contain an endpoint of α and α'. We call α the *intersection arc of P on k* (and α' the *intersection arc of P on c*). The length of either is called the *intersection length* of k and c at P and denoted by $|P|_{\mathcal{E}}$, i.e.:

$$| \, P \, |_{\mathcal{E}} = | \, \alpha \, |_{\mathcal{E}} \; = | \, \alpha' \, |_{\mathcal{E}}$$

Note that, as every pair-of-pants complementary to $\mathcal{E}$ has precisely three boundary curves, the intersection arcs on c and k are well defined by P. Furthermore, one has always $|P|_{\mathcal{E}} \geq 1$ unless P is contained in a subarc of k or c that is a wave with respect to $\mathcal{E}$.

Definition 6.1. Let c be a simple closed curve on a surface Σ of genus $g \geq 2$. Assume that c is tight with respect to some complete decomposing system $\mathcal{E}$ of Σ. An arc c' on the curve c will be called *small* (with respect to $\mathcal{E}$) if

$$|c'|_{\mathcal{E}} \; < \; \frac{3}{12g-11} |c|_{\mathcal{E}} \, .$$

If c' is not small, it will be called *large* (with respect to $\mathcal{E}$).

We now use Definition 2.9 and assume that τ is a maximal train track on the surface Σ which is tight with respect to the complete decomposing system $\mathcal{E}$. We denote by $|\tau|_{\mathcal{E}}$ the *$\mathcal{E}$-length* of τ, by which we mean the total $\mathcal{E}$-length of any set of arcs α_i such that every regular I-fiber is met by only one of the α_i, and precisely once.

If τ' is a train track derived from τ, then the length $|\tau'|_{\mathcal{E}}$ is precisely given by $|\tau|_{\mathcal{E}}$ plus the sum of the lengths $|\sigma_i|_{\mathcal{E}}$ of all of the unzipping paths σ_i used to derive τ' from τ (compare subsection 2.3). Below we will always use the convention that any unzipping path used to derive τ' from τ is oriented from the cusp point of τ towards the cusp point of τ'.

Since τ is maximal, each complementary component is a triangle, so that by Euler characteristic reasons there must be precisely $4g-4$ such triangles. Each triangle gives rise to precisely 3 unzipping paths σ_i, so that altogether we have $12g-12$ unzipping paths σ_i.

Remark 6.2. (a) By the definition of the deriving process, every σ_i must cover τ, so that one has:

$$|\sigma_i|_{\mathcal{E}} \geq |\tau|_{\mathcal{E}}$$

(b) As a consequence, we obtain:

$$|\tau'|_{\mathcal{E}} \; = \; |\tau|_{\mathcal{E}} + \sum_{i=1,\ldots,12g-11} |\sigma_i|_{\mathcal{E}} \; \geq \; (12g-11)|\tau|_{\mathcal{E}}$$

Proposition 6.3. *Let $\mathcal{E}$ be a complete decomposing system of Σ, let $\tau \supset \tau' \supset \tau'' \supset \tau'''$ be a tower of derived train tracks that are tight with respect to $\mathcal{E}$, and let c be a simple closed curve that is carried by τ'''. Let c' be an arc on c that is large with respect to $\mathcal{E}$. Then one has:*

1. *The arc c' contains a subarc that is parallel with respect to $\mathcal{E}$ to one of the unzipping paths σ_i used to derive τ' from τ.*
2. *The arc c' covers τ.*
3. *Let $\mathcal{D} = \{D_1, \ldots, D_{3g-3}\}$ be a second complete decomposing system on Σ which is tight with respect to $\mathcal{E}$, and assume that for some intersection point $P \in D_i \cap c$ the intersection arc on c is large. Then $\mathcal{D}$ can be carried by τ.*

Proof. (1) We first observe that, since c is carried by τ''', it follows from Lemma 2.4 that c covers τ''.

Since c' is large, we can decompose c' as a concatenation $c' = c_1 \circ c_2 \circ c_3$ where for each $k \in \{1, 2, 3\}$, c_k has length $|c_k|_{\mathcal{E}} \geq \frac{1}{r+1}|c|_{\mathcal{E}}$. Since c covers τ'', we have $|c|_{\mathcal{E}} \geq |\tau''|_{\mathcal{E}}$. Hence Remark 6.2 (b) yields for any $k \in \{1, 2, 3\}$:

$$|c_k|_{\mathcal{E}} \geq \frac{1}{12g-11}|c|_{\mathcal{E}} \geq \frac{1}{12g-11}|\tau''|_{\mathcal{E}} > |\tau'|_{\mathcal{E}}$$

In particular, c_2 must intersect at least one singular fiber I_0 of τ', say in a point Q, and there is at least one of the unzipping paths σ_i which has its terminal point on I_0. Let c'' be the arc on c that starts at Q and runs in the same direction as $\overline{\sigma}_i$ ($= \sigma_i$ with reverted orientation), and has the same length as σ_i. From Remark 6.2 (a) and the above inequality we obtain for $k = 1$ or $k = 3$:

$$|c''|_{\mathcal{E}} = |\sigma_i|_{\mathcal{E}} \leq |\tau'|_{\mathcal{E}} \leq |c_k|_{\mathcal{E}}$$

Hence c'' is a subpath of c'.

We now ask whether c'' runs parallel to σ_i on τ' (or on τ). The only way in which this can fail to happen is if at some singular fiber I_1 of τ' the two paths branch off each other on τ'. Let σ_j be the unzipping path with terminal point at I_1. Notice that the two branches of τ'

on either side of σ_j run still parallel on τ. Thus either c'' and σ_i run parallel on τ throughout all of σ_i, or else c'' runs parallel to all of σ_j (or parallel to all of some other σ_k, which we then rename σ_j for the rest of the proof), before branching off σ_i on τ. Thus c'' runs parallel on τ for the entire length of either σ_i, or for the entire length of σ_j. Since τ is tight with respect to $\mathcal{E}$, the same assertion is true with "parallel on τ" replaced by "parallel with respect to $\mathcal{E}$". Hence assertion (1) of the lemma is proved.

(2) This is a direct consequence of the proof given above for assertion (1) since, by definition, any of the unzipping paths σ_i (and hence also any path parallel to σ_i on τ) covers τ.

(3) The intersection arc α' at P is an arc on c, and since it is large, it follows from assertion (2) that it covers τ. The arc α on D_i which is parallel to α' (compare the paragraph before Definition 6.1) can be isotoped close to α' and thus to an arc which is parallel to α' on τ, so that it also covers τ. This isotopy preserves the property that $\mathcal{D}$ is tight with respect to $\mathcal{E}$, as by definition (see the beginning of this section) it is an isotopy of the pair $(\Sigma, \mathcal{E})$. We can thus apply Proposition 2.11 and obtain that $\mathcal{D}$ is carried by τ.

$\square$

Proposition 6.4. *Let $\mathcal{E} \subset \Sigma$ be a complete decomposing system and let c and D be simple closed essential curves on Σ which are tight with respect to $\mathcal{E}$. We also assume that for any intersection point $P \in D \cap c$ the intersection length $|P|_{\mathcal{E}}$ is small.*

Let $\delta_c^m(D)$ denote the curve obtained from D by an m-fold Dehn twist at c and subsequent tightening with respect to $\mathcal{E}$. Then for any non-trivial twist exponent $m \in \mathbb{Z} \smallsetminus \{0\}$ and any intersection point $S \in \delta_c^m(D) \cap c$ the intersection length $|S|_{\mathcal{E}}$ is large.

Proof. At every intersection point $P \in D \cap c$ we perform the m-fold Dehn twist at P in a two-step procedure as follows:

Step 1: Choose a small embedded annulus neighborhood A_P of c in Σ. The point P is contained in the arc $\beta = A_P \cap D$. After a suitable isotopy of D we can assume that β is entirely contained in one of the pair-of-pants complementary to $\mathcal{E}$. Denote the points in $\partial\beta$ by P^{in} and P^{tr} ("initial" and "terminal" points). Remove the arc β from D and instead insert an arc $\eta \subset A_P$ with $\partial\eta = \{P^{in}, P^{tr}\}$, where η winds around m times around the core curve of A_P in the direction determined according to whether $m > 0$ or $m < 0$. After a suitable isotopy we can assume that the arc η meets c only in a single point S.

Step 2: Now perform an isotopy of the new curve $\widehat{D} = (D \smallsetminus \beta) \cup \eta$, which tightens it with respect to $\mathcal{E}$, in order to obtain the curve $\delta_c^m(D)$. This is done by isotoping off $\mathcal{E}$ two pairs of parallel arcs: The arcs in the first pair are concatenated at P^{in} and the arcs in the second pair are concatenated at P^{tr}. In each pair one of the arcs lies on η, and the other on $D \smallsetminus \beta$. The tightening isotopy will move the points P^{in}, P^{tr} in opposite directions along paths determined by each pair of parallel arcs.

There are now several cases to be considered, and in order to do so in a precise way, we introduce the following notation for the arcs in D that will be cancelled as described in the above step 2:

Consider the intersection arc α of P on D. Since c and D are tight with respect to $\mathcal{E}$, the point P lies on the arc $\widehat{\alpha}$ which is the prolongation of α into the pairs-of-pants adjacent to the curves of $\mathcal{E}$ containing $\partial\alpha$. (Recall that $\partial\alpha \subset \mathcal{E}$.) Denote by $\widehat{D}^{in}$ and $\widehat{D}^{tr}$ the two connected components of $\alpha - \beta$. Recall that all of β and hence P, P^{in} and P^{tr} are contained in the interior of one of the pair-of-pants complementary to $\mathcal{E}$. In case where β is contained in $\widehat{\alpha} \smallsetminus \alpha$, then one of the two, $\widehat{D}^{in}$ or $\widehat{D}^{tr}$, is empty. Similarly, denote by η^{in} and η^{tr} the maximal initial and terminal subarcs of η which run parallel to $\widehat{D}^{in}$ and $\widehat{D}^{tr}$ respectively. Note that the arcs η^{in} and $\widehat{D}^{in}$ are concatenated at P^{in} and the arcs η^{tr} and $\widehat{D}^{tr}$ are concatenated at P^{tr}. Note also that both pairs of concatenated arcs can be cancelled by an isotopy that moves the points P^{in} and P^{tr} in opposite directions along $\widehat{\alpha}$. Furthermore, note the following crucial fact:

$(*)$ By the definition of the intersection arc α the cancelling isotopy can not be extended further, as the extension of the above arcs, along η and D respectively, must leave each of the two pairs-of-pants adjacent to α through distinct boundary curves.

If there is only a single intersection point P of D with c, then the above paragraph gives a precise description of an m-fold Dehn-twist of D along c. However, if there are several such intersection points, one has to be much more careful: The difficulty comes from the fact that for distinct intersection points the above subarcs may overlap, so that it becomes impossible to perform the above tightening isotopies at all intersection points at the same time. One also needs an argument to show that after performing at adjacent intersection points two such isotopies which end in the same pair-of-pants complementary to $\mathcal{E}$, there is no possibility to continue the tightening process further.

In order to analyse the cases systematically, we first observe that for a single intersection point P the subarcs η^{in} and η^{tr} of η can not overlap (on η): This is due to the hypothesis that the intersection length at P is small, since both η^{in} and η^{tr} must run parallel to a subarc of the intersection arc at P, while the length of η is a multiple of the length of c.

Hence for each point P we can choose a point $\overline{P}$ on $\mathcal{E} \cap \eta$ which is not contained in either η^{in} or η^{tr}. We will now show that, if P and P' are subsequent intersection points on D, then the segment $[\overline{P}, \overline{P}'] \subset \widehat{D}$ between the corresponding points $\overline{P}$ and $\overline{P}'$ on $\widehat{D}$ becomes tight after cancelling the segments η^{in}, η^{tr}, $\widehat{D}^{in}$ and $\widehat{D}^{tr}$, or subsegments of the latter, but that no further cancelation of intersection points with $\mathcal{E}$ ever occurs in the process of tightening the segment $[\overline{P}, \overline{P}']$.

Note that, if the intersection length at P is bigger than 0, one can isotope the intersection point P, and with it the arc β, along the intersection arc of P on c. Such an isotopy gives rise to a "trade-off" between the lengths of η^{in} and $\widehat{D}^{in}$ on one hand, and η^{tr} and $\widehat{D}^{tr}$ on the other. Of course, the total number of possible cancellations is not affected by such a trade-off move. Furthermore, the corresponding point $\overline{P}$ introduced above can be kept fixed while performing a trade-off move at P.

As before, let P and P' two intersection points that are adjacent on D, and let $\widehat{d}$ denote the segment of D which lies between the subarcs β and β' corresponding as above to P and P'. Since P and P' are adjacent, $\widehat{d}$ does not contain another intersection point of D with c. Thus $\widehat{d}$ can also be viewed as subarc of $\widehat{D}$: It is precisely the segment between the subarcs η and η'. We distinguish three cases:

(1) The intersection arcs of P and P' on D do not overlap along $\widehat{d}$, and furthermore they are separated on $\widehat{d}$ by at least one intersection point, say R, of $D \cap \mathcal{E}$. Thus in this case, the corresponding cancellation arcs on $\widehat{D}$, say $\widehat{D}^{tr}$ and $\widehat{D'}^{in}$, cannot overlap on $\widehat{d}$, as they are separated by R. Hence, by the fact $(*)$ stated above, after canceling $\widehat{D}^{tr}$ against η^{tr} and $\widehat{D'}^{in}$ against η'^{in} the resulting segment between $\overline{P}$ and $\overline{P}'$ will be tight (see Figure 1).

(2) The intersection arcs of P and P' on $\widehat{D}$ do overlap along $\widehat{d}$. Hence we can perform a trade-off move as defined above, so that after this isotopy both, P and P', together with the small corresponding arcs β and β', all come to lie in the same complementary pair-of-pants $\mathcal{P}$, and $\widehat{d}$ becomes a very small arc contained in $\mathcal{P}$ (and thus disjoint from

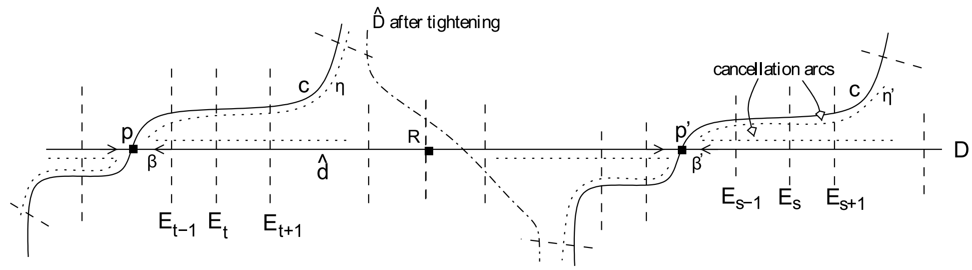

FIGURE 1. The intersection arcs do not overlap over $\widehat{d}$.

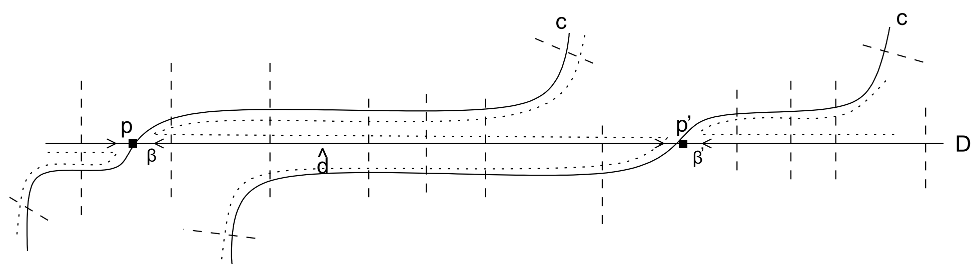

FIGURE 2. The intersection arcs do overlap over $\widehat{d}$.

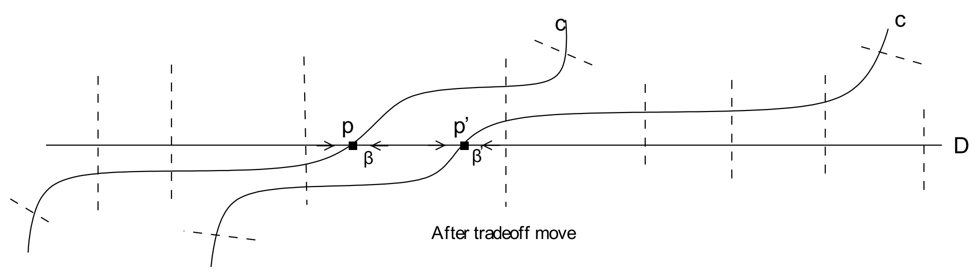

FIGURE 3. Itersection arcs after the "trade off".

$\mathcal{E}$). Note that this can be done only if the annulus neighborhood A of c was chosen sufficiently thin. Now observe that the directions of the two inserted arcs η and η' coincide, as the "twisting direction" of a Dehn twist is well defined and independent of the local orientation of the curve c: This is a well known fact for Dehn twists.

As a consequence, in this second case no cancellation at all is possible on the segment of $\widehat{D}$ between $\overline{P}$ and $\overline{P}'$: This arc is parallel to a concatenation of $\widehat{d}$ with two arcs that each winds around c at most m times, and thus is already tight as is (see Figures 2 and 3).

(3) The intersection arcs of P and P' on $\widehat{D}$ do not overlap along $\widehat{d}$, but they have endpoints Q and Q' which are contained in curves $E_i, E_j \in \mathcal{E}$, and Q, Q' are adjacent points of $\widehat{D} \cap \mathcal{E}$ on $\widehat{d}$:

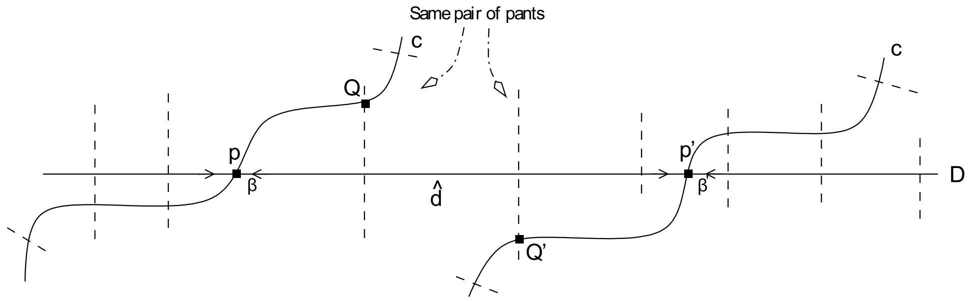

FIGURE 4. End points Q and Q' on boundary curves of the same pair of pants .

This implies that there is a pair-of-pants $\mathcal{P}$ complementary to $\mathcal{E}$ which contains both, Q and Q', but on distinct boundary curves of $\mathcal{P}$. In this case we can again perform a trade-off move, so that P and P' move along $\widehat{d}$ beyond Q and Q' respectively, and both come to lie inside of $\mathcal{P}$.

But then the same argument, as in case (2) above, applies: The arcs β and β' can also be assumed to lie in $\mathcal{P}$, the segment $\widehat{d}$ degenerates to a small arc entirely contained in $\mathcal{P}$, and the segment on $\widehat{D}$ between $\overline{P}$ and $\overline{P}'$ is tight as is, without any cancellations at either $\widehat{P}^{tr}$ or $\widehat{P}'^{in}$ (see Figure 4).

Hence in each of the three possible cases the tightening process needed to obtain $\delta_c^m(D)$ from $\widehat{D}$ is complete if at any one of the intersection points $P \in D \cap \mathcal{E}$ an isotopy is performed which cancels η^{in} against $\widehat{D}^{in}$ and η^{tr} against $\widehat{D}^{tr}$, or subarcs of those.

However, by definition the intersection arc α of P on D is equal to the concatenation $\widehat{D}^{in} \circ \beta \circ \widehat{D}^{tr}$, and β is disjoint from $\mathcal{E}$. Hence one obtains $|P|_{\mathcal{E}} = |\alpha|_{\mathcal{E}} = |\widehat{D}^{in}|_{\mathcal{E}} + |\widehat{D}^{tr}|_{\mathcal{E}} = |\eta^{in}|_{\mathcal{E}} + |\eta^{tr}|_{\mathcal{E}}$.

On the other hand, all of the arc η runs parallel to an arc on c, and one has $|\eta|_{\mathcal{E}} = m|c|_{\mathcal{E}}$. Hence the intersection length of $\delta_c^m(D)$ at S is bounded below by $|\eta \smallsetminus (\eta^{in} \cup \eta^{tr})|_{\mathcal{E}} = |\eta|_{\mathcal{E}} - (|\eta^{in}|_{\mathcal{E}} + |\eta^{tr})|_{\mathcal{E}})$. For $|m| \geq 1$ we obtain:

$$|S|_{\mathcal{E}} \;\geq\; |\eta|_{\mathcal{E}} - (|\eta^{in}|_{\mathcal{E}} + |\eta^{tr})|_{\mathcal{E}}) \;\geq\; |m| \cdot |c|_{\mathcal{E}} - |P|_{\mathcal{E}}$$

$$> \; |c|_{\mathcal{E}} - \frac{3}{12g-11}|c|_{\mathcal{E}} \; > \; \frac{r-2}{12g-11}|c|_{\mathcal{E}} \; > \; \frac{12g-14}{12g-11}|c|_{\mathcal{E}} \; > \; \frac{3}{12g-11}|c|_{\mathcal{E}}$$

This shows that the intersection length of $\delta_c^m(D)$ at any of its intersection points with $\mathcal{E}$ is large.

□

Corollary 6.5. *Let $\mathcal{E} \subset \Sigma$ be a complete decomposing system and let c and D be simple closed essential curves on Σ which are tight with respect to $\mathcal{E}$. We also assume that for every intersection point $P \in D \cap c$ the intersection length $|P|_{\mathcal{E}}$ is small.*

Let $\delta_c^m(D)$ denote the curve obtained from D by an m-fold Dehn twist at c and subsequent tightening with respect to $\mathcal{E}$. Then the total number of intersections of $\delta_c^m(D)$ with $\mathcal{E}$ achieves a strict minimum at $m = 0$, where $m \in \mathbb{Z}$ is the twist exponent:

$$\#\delta_c^m(D) \cap \mathcal{E} \;>\; \#D \cap \mathcal{E} \qquad \text{for all} \qquad m \neq 0$$

Proof. This is a direct consequence of the fact, shown in detail in the last proof, that the curves D and $\delta_c^m(D)$ differ only along the intersection arcs of their intersection points with c, and for those the statement of Proposition 6.4 gives directly the desired inequality.

□

We will now go back to the setting of subsection 2.7: Let M be a closed 3-manifold with Heegaard splitting $M = V \cup_\Sigma W$, and let $n \in \mathbb{N}$. A curve c on the Heegaard surface Σ is called *n-optimal* if for any $m \in \mathbb{Z} \smallsetminus \{0\}$ one has

$$d(V, W_c^m) \geq n\,,$$

where the handlebody W_c^m is obtained from W by composing the attaching map $\partial W \to \Sigma$ with an m-fold Dehn twist along c (compare the discussion at the beginning of section 1). The induced Heegaard splitting (V, W_c^m) defines a 3-manifold $M_c^m = V \cup_\Sigma W_c^m$ which can be obtained alternatively by horizontal surgery at $c \subset \Sigma$, see Remark 1.2.

Definition 6.6. A simple closed curve c on Σ is called *n-gregarious* with respect to some complete decomposing system $\mathcal{E}$ on Σ, for any integer $n \geq 0$, if c is n-gregarious with respect to some fat train track τ on Σ with $\mathcal{E}_\tau = \mathcal{E}$.

Theorem 6.7. *Let M be an oriented 3-manifold with a Heegaard splitting $M = V \cup_\Sigma W$, and let $n \in \mathbb{N}$ be an integer that satisfies $n > d(V, W)$. Consider any complete decomposing systems $\mathcal{D} \in \mathcal{CDS}(V)$ and $\mathcal{E} \in \mathcal{CDS}(W)$. Then any essential simple closed curve c on Σ that is $(n+3)$-gregarious with respect to both, $\mathcal{D}$ and $\mathcal{E}$, is n-optimal.*

Proof. (a) We first use Corollary 6.5 to argue that if for some exponent $m_0 \in \mathbb{Z}$ the intersection length $|P|_{\mathcal{E}}$ is small, at every intersection point $P \in D_i \cup c$ and for every $D_i \in \delta_c^{m_0}(\mathcal{D})$, then the total number of intersection points satisfies

$$\#(\delta_c^m(\mathcal{D}) \cap \mathcal{E}) \ > \ \#(\delta_c^{m_0}(\mathcal{D}) \cap \mathcal{E})$$

for all exponents $m \neq m_0$. Since the intersection number is invariant under homeomorphisms, we can apply δ_c^{-m} to obtain

$$\#(\delta_c^m(\mathcal{D}) \cap \mathcal{E}) \ = \ \#(\mathcal{D} \cap \delta_c^{-m}(\mathcal{E}))$$

for all $m \in \mathbb{Z}$. From the symmetry of the conditions in $\mathcal{D}$ and $\mathcal{E}$ we deduce that, if for some exponent $m_1 \in \mathbb{Z}$ the intersection length $|Q|_{\mathcal{D}}$ is small, at every intersection point $Q \in E_j \cup c$ and for every $E_j \in \delta_c^{m_1}(\mathcal{E})$, then one must have:

$$m_0 = m_1$$

(b) We now consider a situation where for some $m \in \mathbb{Z}$ there are intersection points $P \in D_i \cup c$ for some $D_i \in \delta_c^m(\mathcal{D})$ and $Q \in E_j \cup c$ for some $E_j \in \delta_c^m(\mathcal{E})$ such that both, $|P|_{\mathcal{E}}$ and $|Q|_{\mathcal{D}}$, are large. In this case we can apply Proposition 6.3 (c) to deduce that $\mathcal{D}$ and $\mathcal{E}$ are n-gregarious with respect to each other, and hence in particular they don't have waves with respect to each other. Thus we can apply Theorem 2.16 to deduce that $d(V, W_c^m) \geq n$. As a consequence, it follows from the hypothesis $n > d(V, W)$ that $m \neq 0$.

(c) It follows from the above part (a) that there is at most one value $m_0 \in \mathbb{Z}$ for which no point P as in the hypothesis of part (b) exists, and at most one $m_1 \in \mathbb{Z}$ without an analogous point Q. Furthermore, from (a) we know that if there exist both, m_0 and m_1, then one has $m_0 = m_1$. In any case, there is at most one exceptional value $m' = m_0, m' = m_1$ or $m' = m_0 = m_1$, such that the hypotheses of part (b) are given for all integers $m \neq m'$. Thus we conclude from part (b) that $d(V, W_c^m) \geq n$ for all $m \in \mathbb{Z} \smallsetminus \{m'\}$, and that hence that $m' = 0$. This proves that the curve c is n-optimal.

□

Theorem 6.8. *Let M be a closed 3-manifold with Heegaard splitting $M = V \cup_\Sigma W$. For any $n \geq 1$ set of $\mathcal{C}_n^M(\Sigma)$ of n-optimal curves is generic in the set $\mathcal{C}^0(\Sigma)$ of all essential simple closed curves on Σ.*

Proof. We first observe that for any integer $n' \geq n$ one has $\mathcal{C}_{n'}^M(\Sigma) \subset \mathcal{C}_n^M(\Sigma)$, by the definition of n-optimal. Thus, by Remark 4.4 (b), it suffices to prove the statement for $n > d(V, W)$.

We apply Lemma 2.15 to find $\mathcal{D}$ and $\mathcal{E}$ as in Theorem 6.7, and deduce from the latter that the set $\mathcal{C}^M_n(\Sigma)$ contains the intersection of the set $\mathbf{G^{n+3}}(V)$ and the set $\mathbf{G^{n+3}}(W)$. According to Proposition 5.1 both of these sets are generic in $\mathcal{C}^0(\Sigma)$. Hence part (a) of Remark 4.4 shows that the intersection is generic, and part (b) implies that the set $\mathcal{C}^M_n(\Sigma)$ is generic in $\mathcal{C}^0(\Sigma)$.

□

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